International Journal of Mechanical and Industrial Technology ISSN 2348-7593 (Online)
Vol. 12, Issue 2, pp: (1-5), Month: October 2024 - March 2025, Available at: www.researchpublish.com

Fractional Derivatives of Some Fractional Functions

Chii-Huei Yu

School of Mathematics and Statistics, Zhaoqing University, Guangdong, China

DOI: https://doi.org/10.5281/zenodo.13969072

Published Date: 22-October-2024

Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional derivative and a new multiplication of fractional analytic functions, we find arbitrary order fractional derivative of two fractional functions. Fractional binomial theorem plays an important role in this study. In fact, our results are generalizations of the results in classical calculus.**

Keywords: **Jumarie type of R-L fractional derivative, new multiplication, fractional analytic functions, fractional binomial theorem.**

I. INTRODUCTION

Fractional calculus deals with the derivatives and integrals of any real or complex order. In recent years, fractional calculus has been widely popularized and valued because of its applications in various fields such as mechanics, dynamics, elasticity, electronics, physics, modeling, economics, and control theory [1-8]. Fractional calculus is different from classical calculus. There is no unique definition of fractional derivative and integral. Commonly used definitions include Riemann Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald Letnikov (G-L) fractional derivative, conformable fractional derivative, and Jumarie's modified R-L fractional derivative [9-11]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie type of R-L fractional derivative, we obtain arbitrary order fractional derivative of the following two fractional functions:

$$sin_\alpha\left(cosh_\alpha(x^\alpha)\right)$$

and

$$cos_\alpha\left(sinh_\alpha(x^\alpha)\right)$$

A new multiplication of fractional analytic functions and fractional binomial theorem play important roles in this paper. In fact, our results are generalizations of classical calculus results.

II. PRELIMINARIES

At first, we introduce the fractional derivative used in this paper and its properties.

Definition 2.1 ([12])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^\alpha\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^\alpha}dt\,, \tag{1}$$

where $\Gamma(\ \ )$ is the gamma function. On the other hand, for any positive integer n, we define $\left({}_{x_0}D_x^\alpha\right)^n[f(x)] = \left({}_{x_0}D_x^\alpha\right)\left({}_{x_0}D_x^\alpha\right)\cdots\left({}_{x_0}D_x^\alpha\right)[f(x)]$, the n-th order α-fractional derivative of $f(x)$.

Proposition 2.2 ([13])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{3}$$

Definition 2.3 ([14])**:** Let x, x_0 and a_k be real numbers for all k, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a, b] \to R$ can be expressed as $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at $x = x_0$.

Next, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([15])**:** If $0 < \alpha \leq 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}, \tag{4}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}. \tag{5}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \otimes_\alpha \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha} \\ &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left(\sum_{m=0}^{k}\binom{k}{m} a_{k-m} b_m\right)(x-x_0)^{k\alpha}. \end{aligned} \tag{6}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{k=0}^{\infty} \frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k} \otimes_\alpha \sum_{k=0}^{\infty} \frac{b_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{m=0}^{k}\binom{k}{m} a_{k-m} b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha k}. \end{aligned} \tag{7}$$

Definition 2.5 ([16])**:** Assume that $0 < \alpha \leq 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \tag{8}$$

And the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2k}, \tag{9}$$

and

$$sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2k+1)}. \tag{10}$$

On the other hand, the α-fractional hyperbolic cosine function is defined by

$$cosh_\alpha(x^\alpha) = \frac{1}{2}\left[E_\alpha(x^\alpha) + E_\alpha(-x^\alpha)\right]. \tag{11}$$

The α-fractional hyperbolic sine function is defined by

$$sinh_\alpha(x^\alpha) = \frac{1}{2}\left[E_\alpha(x^\alpha) - E_\alpha(-x^\alpha)\right]. \tag{12}$$

Theorem 2.6 (fractional binomial theorem)**:** *If* $0<\alpha\leq 1$, p *is a positive integer and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ *are two* α-*fractional analytic functions. Then*

$$[f_\alpha(x^\alpha)+g_\alpha(x^\alpha)]^{\otimes_\alpha p}=\sum_{k=0}^{p}\binom{p}{k}\left(f_\alpha(x^\alpha)\right)^{\otimes_\alpha k}\otimes_\alpha\left(g_\alpha(x^\alpha)\right)^{\otimes_\alpha (p-k)}\ , \tag{13}$$

where $\binom{p}{k}=\frac{p!}{k!(p-k)!}$.

III. MAIN RESULTS

In this section, we find arbitrary order fractional derivative of two fractional functions: $sin_\alpha\left(cosh_\alpha(x^\alpha)\right)$ and $cos_\alpha\left(sinh_\alpha(x^\alpha)\right)$.

Theorem 3.1: *If* $0<\alpha\leq 1$ *and* n *is a positive integer. Then*

$$\left({}_0D_x^\alpha\right)^n\left[sin_\alpha\left(cosh_\alpha(x^\alpha)\right)\right]=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!\cdot 2^{2k+1}}\left[\sum_{m=0}^{2k+1}\binom{2k+1}{m}(2m-2k-1)^n E_\alpha((2m-2k-1)x^\alpha)\right]. \tag{14}$$

Proof By fractional binomial theorem,

$$sin_\alpha\left(cosh_\alpha(x^\alpha)\right)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(cosh_\alpha(x^\alpha)\right)^{\otimes_\alpha (2k+1)}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{1}{2}[E_\alpha(x^\alpha)+E_\alpha(-x^\alpha)]\right)^{\otimes_\alpha (2k+1)}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!\cdot 2^{2k+1}}([E_\alpha(x^\alpha)+E_\alpha(-x^\alpha)])^{\otimes_\alpha (2k+1)}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!\cdot 2^{2k+1}}\left[\sum_{m=0}^{2k+1}\binom{2k+1}{m}\left(E_\alpha(x^\alpha)\right)^{\otimes_\alpha m}\otimes_\alpha\left(E_\alpha(-x^\alpha)\right)^{\otimes_\alpha (2k+1-m)}\right]$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!\cdot 2^{2k+1}}\left[\sum_{m=0}^{2k+1}\binom{2k+1}{m}E_\alpha(mx^\alpha)\otimes_\alpha E_\alpha(-(2k+1-m)x^\alpha)\right]$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!\cdot 2^{2k+1}}\left[\sum_{m=0}^{2k+1}\binom{2k+1}{m}E_\alpha((2m-2k-1)x^\alpha)\right]. \tag{15}$$

Thus,

$$\left({}_0D_x^\alpha\right)^n\left[sin_\alpha\left(cosh_\alpha(x^\alpha)\right)\right]$$

$$=\left({}_0D_x^\alpha\right)^n\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!\cdot 2^{2k+1}}\left[\sum_{m=0}^{2k+1}\binom{2k+1}{m}E_\alpha((2m-2k-1)x^\alpha)\right]\right]$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!\cdot 2^{2k+1}}\left[\sum_{m=0}^{2k+1}\binom{2k+1}{m}\left({}_0D_x^\alpha\right)^n[E_\alpha((2m-2k-1)x^\alpha)]\right]$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!\cdot 2^{2k+1}}\left[\sum_{m=0}^{2k+1}\binom{2k+1}{m}(2m-2k-1)^n E_\alpha((2m-2k-1)x^\alpha)\right].$$ q.e.d.

Theorem 3.2: *If* $0<\alpha\leq 1$ *and* n *is a positive integer. Then*

$$\left({}_0D_x^\alpha\right)^n\left[cos_\alpha\left(sinh_\alpha(x^\alpha)\right)\right]=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!\cdot 2^{2k}}\left[\sum_{m=0}^{2k}(-1)^m\binom{2k}{m}(2m-2k)^n E_\alpha((2m-2k)x^\alpha)\right]. \tag{16}$$

Proof Using fractional binomial theorem yields

$$cos_\alpha\left(sinh_\alpha(x^\alpha)\right)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left(sinh_\alpha(x^\alpha)\right)^{\otimes_\alpha 2k}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left(\frac{1}{2}[E_\alpha(x^\alpha)-E_\alpha(-x^\alpha)]\right)^{\otimes_\alpha 2k}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!\cdot 2^{2k}}([E_\alpha(x^\alpha)-E_\alpha(-x^\alpha)])^{\otimes_\alpha 2k}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!\cdot 2^{2k}}\left[\sum_{m=0}^{2k}\binom{2k}{m}\left(E_\alpha(x^\alpha)\right)^{\otimes_\alpha m}\otimes_\alpha\left(-E_\alpha(-x^\alpha)\right)^{\otimes_\alpha (2k-m)}\right]$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!\cdot 2^{2k}}\left[\sum_{m=0}^{2k}\binom{2k}{m}(-1)^{2k-m}E_\alpha(mx^\alpha)\otimes_\alpha E_\alpha(-(2k-m)x^\alpha)\right]$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!\cdot 2^{2k}}\left[\sum_{m=0}^{2k}(-1)^m\binom{2k}{m}E_\alpha((2m-2k)x^\alpha)\right]. \tag{17}$$

Therefore,

$$\left({}_0D_x^\alpha\right)^n\left[\cos_\alpha\left(\sinh_\alpha(x^\alpha)\right)\right]$$

$$=\left({}_0D_x^\alpha\right)^n\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!\cdot 2^{2k}}\left[\sum_{m=0}^{2k}(-1)^m\binom{2k}{m}E_\alpha((2m-2k)x^\alpha)\right]\right]$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!\cdot 2^{2k}}\left[\sum_{m=0}^{2k}(-1)^m\binom{2k}{m}\left({}_0D_x^\alpha\right)^n\left[E_\alpha((2m-2k)x^\alpha)\right]\right]$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!\cdot 2^{2k}}\left[\sum_{m=0}^{2k}(-1)^m\binom{2k}{m}(2m-2k)^n E_\alpha((2m-2k)x^\alpha)\right].$$ q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional derivative, we obtain arbitrary order fractional derivative of two fractional functions. A new multiplication of fractional analytic functions and fractional binomial theorem play important roles in this study. In fact, our results are generalizations of the results in ordinary calculus. In the future, we will continue to use the fractional analytic functions method to solve the problems in engineering mathematics and fractional differential equations.

REFERENCES

[1] E. Soczkiewicz, Application of fractional calculus in the theory of viscoelasticity, Molecular and Quantum Acoustics, vol.23, pp. 397-404. 2002.

[2] F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics, Fractals and Fractional Calculus in Continuum Mechanics, A. Carpinteri and F. Mainardi, Eds., pp. 291-348, Springer, Wien, Germany, 1997.

[3] V. E. Tarasov, Mathematical economics: application of fractional calculus, Mathematics, vol. 8, no. 5, 660, 2020.

[4] R. Almeida, N. R. Bastos, and M. T. T. Monteiro, Modeling some real phenomena by fractional differential equations, Mathematical Methods in the Applied Sciences, vol. 39, no. 16, pp. 4846-4855, 2016.

[5] Mohd. Farman Ali, Manoj Sharma, Renu Jain, An application of fractional calculus in electrical engineering, Advanced Engineering Technology and Application, vol. 5, no. 2, pp, 41-45, 2016.

[6] W. S. Chung, Fractional Newton mechanics with conformable fractional derivative, Journal of Computational and Applied Mathematics, vol. 290, pp. 150-158, 2015.

[7] S. Das, Functional Fractional Calculus, 2nd Edition, Springer-Verlag, 2011.

[8] M. F. Silva, J. A. T. Machado, A. M. Lopes, Fractional order control of a hexapod robot, Nonlinear Dynamics, vol. 38, pp. 417-433, 2004.

[9] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, Calif, USA, 1999.

[10] K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations; John Willy and Sons, Inc.: New York, NY, USA, 1993.

[11] K. B. Oldham, J. Spanier, The Fractional Calculus; Academic Press: New York, NY, USA, 1974. [12]. F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechanics, Fractals and Fractional Calculus in Continuum Mechanics, A. Carpinteri and F. Mainardi, Eds., pp. 291-348, Springer, Wien, Germany, 1997.

[12] C. -H. Yu, Using integration by parts for fractional calculus to solve some fractional integral problems, International Journal of Electrical and Electronics Research, vol. 11, no. 2, pp. 1-5, 2023.

[13] C. -H. Yu, Infinite series expressions for the values of some fractional analytic functions, International Journal of Interdisciplinary Research and Innovations, vol. 11, no. 1, pp. 80-85, 2023.

[14] C. -H. Yu, Study of fractional analytic functions and local fractional calculus, International Journal of Scientific Research in Science, Engineering and Technology, vol. 8, no. 5, pp. 39-46, 2021.

[15] C. -H. Yu, Exact solutions of some fractional power series, International Journal of Engineering Research and Reviews, vol. 11, no. 1, pp. 36-40, 2023.

[16] C. -H. Yu, Research on a fractional exponential equation, International Journal of Novel Research in Interdisciplinary Studies, vol. 10, no. 1, pp. 1-5, 2023.